AN (FA)-GROUP THAT IS NOT (F$\mathbb{R}$)

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Abstract. An example is given of a finitely generated group L that has a non-trivial action on an $\mathbb{R}$-tree but which cannot act, without fixing a vertex, on any simplicial tree. Moreover, any finitely presented group mapping onto L does have a fixed point-free action on some simplicial tree.

1. Introduction

In [17, p. 286] Peter Shalen asked the following:

Question A. Suppose that Γ is a finitely generated group which admits a non-trivial action by isometries on some $\mathbb{R}$-tree. Does it then follow that Γ admits a non-trivial action, by isometries and without inversions, on a $\mathbb{Z}$-tree? Equivalently, does Γ admit a non-trivial decomposition as the fundamental group of a graph of groups?

In this paper we show that the answer to this question is negative by constructing a finitely generated group L that has a non-trivial (i.e., without global fixed points) action on some $\mathbb{R}$-tree T but which has no non-trivial action on any simplicial tree. Recall that a group G is said to satisfy *Serre's property* (FA) if any simplicial action (without edge inversions) of G on a simplicial tree has a global fixed point (see [16, I.6.1]). Similarly, G has *property* (F$\mathbb{R}$) if every isometric action of G on an $\mathbb{R}$-tree fixes a point. The main result of this paper is

Theorem 1.1. *There exists a finitely generated group L which has property* (FA) *but does not have property* (F$\mathbb{R}$). *Moreover, L is not a quotient of any finitely presented group with property* (FA).

In fact, our approach (using Construction 4.2) shows that any finitely generated group G_0 can be embedded in a group L satisfying the claim of Theorem 1.1. Thus there are uncountably many pairwise non-isomorphic groups L with above properties.

The second claim of Theorem 1.1 shows that property (FA) does not define an open subset in the space of marked groups, which answers a question of Yves de Cornulier (see [3] or [18]). This contrasts with the fact that any finitely generated group with property (F$\mathbb{R}$) *is* a quotient of a finitely presented group with this property (this follows from the work of Culler and Morgan [5] and is explicitly stated in [18, Thm. 1.4]).

Recall, that an action of a group Γ on an $\mathbb{R}$-tree is said to be *stable* if there is no sequence of arcs l_i such that l_{i+1} is properly contained in l_i for every i, and for which the stabilizer Γ_i of l_i is properly contained in Γ_{i+1} for every i. In [1] Bestvina and Feighn proved that if a finitely presented group has a non-trivial minimal stable action on an $\mathbb{R}$-tree then it has a non-trivial action on some simplicial tree.

The group L from Theorem 1.1 is not finitely presented and possesses a non-trivial unstable action on some real tree T. It is possible to construct unstable actions of finitely presented groups on $\mathbb{R}$-trees (see [10]), but all of such (known) examples admit non-trivial actions on simplicial trees. It seems probable that the answer to Question A is positive if the group Γ is finitely presented and so we make the following conjecture:

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Conjecture. If a finitely presented group Γ possesses a non-trivial isometric action on an $\mathbb{R}$-tree then Γ also possesses a non-trivial simplicial action (without inversions) on a simplicial tree.

The construction of L uses folding sequences that have been studied in various papers by the first author (for instance in [9]). In [8] he used a folding sequence construction to give a negative answer to another question of Shalen [17, Question D, p. 293], by showing that there is a finitely generated group that has a non-trivial action on an $\mathbb{R}$-tree with finite cyclic arc stabilizers but which has no non-trivial action on a simplicial tree with small arc stabilizers. This example did, however, have a non-trivial action on a simplicial tree with stabilizers which were not small.

In our new examples the arc stabilizers are not small. However the construction depends on finding an ascending sequence of groups that satisfies certain properties (see Section 4). These groups become arc stabilizers in L. It may be possible to construct such a sequence of groups which are small or even finite. The resulting group L would then give a negative answer to both Shalen's questions mentioned above.

2. The Folding Sequence

The definition below describes the families of groups we will be working with.

Definition 2.1. Let $G_0 < G_1 < G_2 < G_3 < \dots$ be a strictly ascending sequence of groups. We will say that this sequence is *good* if for every $i \in \mathbb{N}$ there is an element $a_i \in G_{i+3}$ such that all of the following hold:

(i) a_i centralizes G_{i-1} in G_{i+3};
(ii) $G_{i+3} = \langle G_i, a_i G_i a_i^{-1} \rangle$;
(iii) if G_{i+1} acts on a simplicial tree T then the restriction of this action to G_i fixes a vertex of T.

Suppose that a sequence of groups $G_0 < G_1 < \dots$ satisfies conditions (i) and (ii) of Definition 2.1. Associated with this sequence of groups and elements $a_i \in G_{i+3}$ is an infinite folding sequence, as in Figure 1. More precisely, for every $i \in \mathbb{N}$, let H_{i+2} be a copy of $a_i G_{i+2} a_i^{-1} \leq G_{i+3}$ with a fixed isomorphism $\alpha_i : a_i G_{i+2} a_i^{-1} \to H_{i+2}$. Since $a_i g a_i^{-1} = g$ for all $g \in G_{i-1} < G_{i+1} < G_{i+2}$, the restriction of α_i to $a_i G_{i-1} a_i^{-1} = G_{i-1}$ gives a monomorphism from G_{i-1} to H_{i+2}. Hence we can define the amalgamated free product $L_i = G_{i+1} *_{G_{i-1}} H_{i+2}$, by the presentation

$$L_i := \langle G_{i+1}, H_{i+2} \,\|\, g = \alpha_i(g) \text{ for all } g \in G_{i-1} \rangle.$$

Thus L_i is the fundamental group of the first graph of groups in Figure 1, and L_{i+1} is the fundamental group of the last graph of groups. Let T_i and T_{i+1} be the corresponding Bass-Serre trees. As described in [9] and [8] there is a surjective homomorphism $\phi_i : L_i \to L_{i+1}$ and a morphism also denoted $\phi_i : T_i \to T_{i+1}$ which is equivariant in terms of the group homomorphism ϕ_i. This morphism is a composition of elementary folding operations (the reader unfamiliar with foldings may notice that Type I and II folds below correspond to the transformations of the standard presentations for the fundamental groups of the respective graphs of groups, which can also be done using finite sequences of Tietze moves). In the first vertex morphism, the centre vertex group $\langle G_i, G_i' \rangle$, where $G_i' := \alpha_i(a_i G_i a_i^{-1}) \leqslant H_{i+2}$, is isomorphic to $G_i *_{G_{i-1}} \alpha_i(a_i G_i a_i^{-1})$, so by the universal property of amalgamated free products, there is an epimorphism $\langle G_i, G_i' \rangle \to G_{i+3}$, which is identity on G_i and α_i^{-1} on $G_i' = \alpha_i(a_i G_i a_i^{-1})$ (this is where both properties (i) and (ii) are used). The fundamental group of the graph on the fifth line can obtained from the one for the fourth line by taking a different fundamental region for the action on the corresponding Bass-Serre tree; thus the group is the same but we take a different presentation of it.

The second vertex morphism is the epimorphism $\eta_i : G_{i+1} *_{G_i} (a_i^{-1} H_{i+2} a_i) \to G_{i+2}$ which is the identity on the left hand factor and which uses the natural identification of the right hand factor with G_{i+2} (i.e., $\eta_i(a_i^{-1} g a_i) = a_i^{-1} \alpha_i^{-1}(g) a_i$ for all $g \in H_{i+2}$). Finally, the last line is obtained using the isomorphism $\beta_{i+1} : G_{i+3} \to H_{i+3}$ defined via $\beta_{i+1}(g) = \alpha_{i+1}(a_{i+1} g a_{i+1}^{-1})$ for every $g \in G_{i+3}$.

Repeating this procedure for $i+1, i+2, \dots$ gives an infinite folding sequence and, as described in [9] and [8], there is a limit group L which is the direct limit of the sequence of ϕ_i's and a limit

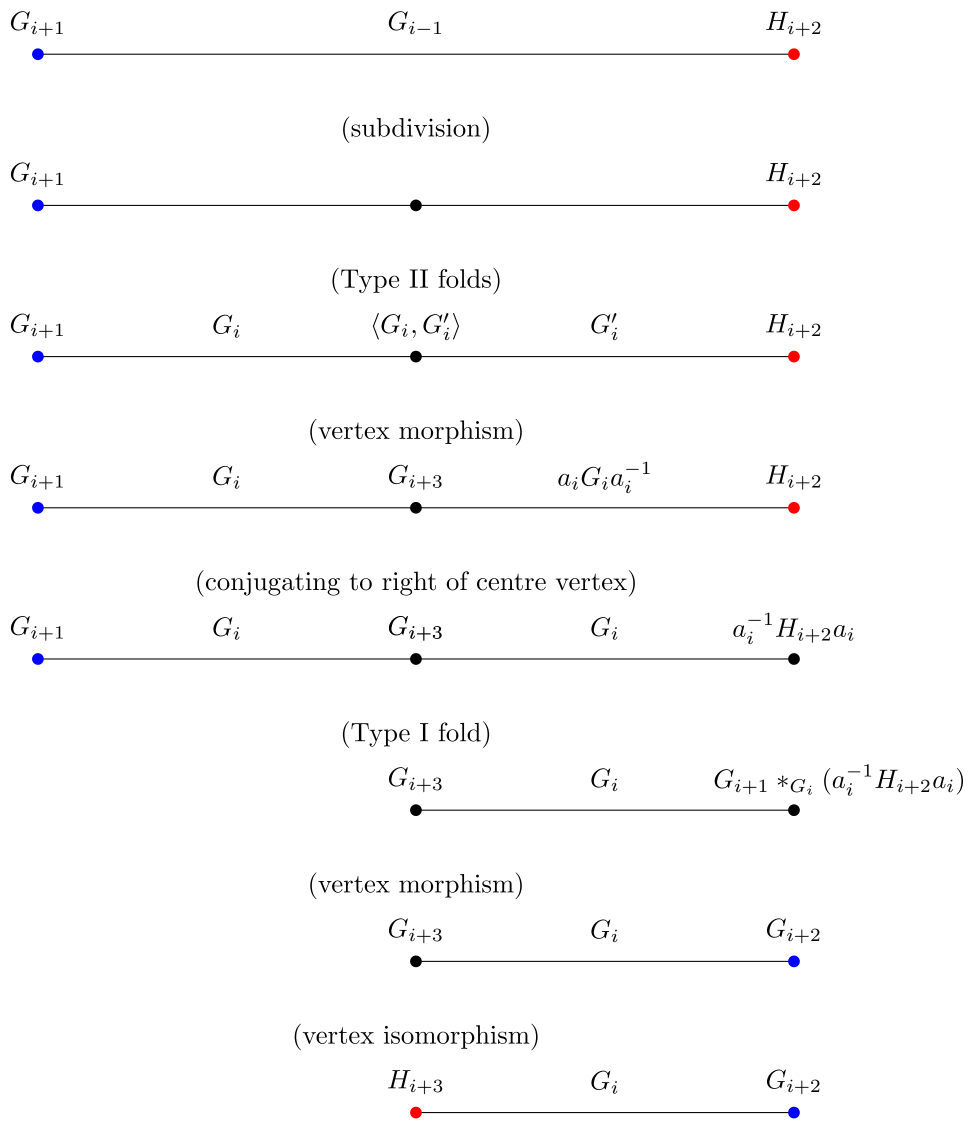

FIGURE 1. Folding sequence of graph of groups

$\mathbb{R}$-tree T on which L acts by isometries. In fact, as in [8], one can put metrics on the trees T_i so that they become simplicial $\mathbb{R}$-trees (also denoted) T_i in such a way that T is a strong limit (in the sense of [12]) of the trees T_i.

Thus the limit group L possesses a non-trivial action on an $\mathbb{R}$-tree. On the other hand, Proposition 3.2 below shows that L has (FA). Some properties of L can be deduced from properties of the G_i. For example, if the G_1 and G_2 are finitely generated, then so is L; if G_i are torsion-free for all i then L is torsion-free too.

We now make several observations regarding the folding homomorphisms described above.

Remark 2.2. For any $i \in \mathbb{N}$ the images of $G_{i+1} \leq L_i$ and $H_{i+2} \leq L_i$ in L_{i+1} have the following properties:

(a) $\phi_i(G_{i+1}) = G_{i+1} < G_{i+2}$ (ϕ_i is the identity on G_{i+1});
(b) $\phi_i(H_{i+2}) = \beta_{i+1}(a_i) G_{i+2} \beta_{i+1}(a_i)^{-1}$ and $\beta_{i+1}(a_i) \in H_{i+3}$;

(c) L_{i+1} is generated by H_{i+3} and $\phi_i(H_{i+2})$.

3. Property (FA) for the limit group L.

As described in the previous section the group L is defined as the direct limit $L = \lim_{i\to\infty}(L_i, \phi_i)$. Thus there is an epimorphism $\psi_i : L_i \to L$ such that

$$\psi_{i+1} \circ \phi_i = \psi_i \text{ for all } i \in \mathbb{N}. \tag{1}$$

In this section we will assume that the sequence $G_0 < G_1 < \dots$ consists of finitely generated groups satisfying the conditions (i)-(iii) from Definition 2.1, and we will prove that any action (simplicial and without inversions) of the limit group L on a simplicial tree has a global fixed vertex.

We introduce some notation: for any $i \in \mathbb{N} \cup \{0\}$ denote $\bar{G}_i := \psi_{i+1}(G_i) \leqslant L$, $\bar{H}_{i+3} := \psi_{i+1}(H_{i+3}) \leqslant L$. Let $b_i := \psi_{i+1}(\beta_{i+1}(a_i)) \in \bar{H}_{i+3}$, $i \in \mathbb{N}$. The property (ii), Remark 2.2 and (1) imply that all of the following hold in the group L:

$$\psi_{i+1} \text{ is injective on } G_i, \text{ consequently } \bar{G}_i \cong G_i,\ i \in \mathbb{N} \cup \{0\}, \tag{2}$$

$$\psi_{i+2}|_{G_i} = \psi_{i+1}|_{G_i}, \text{ thus } \bar{G}_i \leqslant \bar{G}_{i+1} \text{ for all } i \in \mathbb{N} \cup \{0\}, \tag{3}$$

$$\bar{G}_i \leqslant \bar{H}_{i+3},\quad b_i\bar{G}_ib_i^{-1} \leqslant \bar{H}_{i+3} \text{ and } \bar{H}_{i+3} = \langle \bar{G}_i, b_i\bar{G}_ib_i^{-1}\rangle \text{ for all } i \in \mathbb{N}, \tag{4}$$

$$\bar{H}_{i+2} = b_i\bar{G}_{i+2}b_i^{-1} \text{ and } L = \langle \bar{H}_{i+2}, \bar{H}_{i+3}\rangle \text{ for every } i \in \mathbb{N}. \tag{5}$$

Given two vertices x, y of a simplicial tree T, $[x, y]$ will denote the (unique) geodesic segment between x and y in T. If the group L acts on T and $M \subseteq L$, then $\mathrm{Fix}(M)$ will be used to denote the set of all vertices of T that are fixed by M.

Lemma 3.1. *Suppose that L acts on a simplicial tree T, $x \in \mathrm{Fix}(\bar{G}_i)$ and $y \in \mathrm{Fix}(\bar{H}_{i+2})$ for some $i \in \mathbb{N}$. Then $\bar{H}_{i+3}$ fixes some vertex of $[x, y]$.*

Proof. By condition (iii) from Definition 2.1, (2) and (3), $\mathrm{Fix}(\bar{G}_{i+3}) \neq \emptyset$, and (5) implies that $\mathrm{Fix}(\bar{H}_{i+3}) = b_i \circ \mathrm{Fix}(\bar{G}_{i+3}) \neq \emptyset$. Thus there exists some $z \in \mathrm{Fix}(\bar{H}_{i+3})$. Since any geodesic triangle on a tree is a tripod, the intersection $[x, y] \cap [x, z] \cap [y, z]$ consists of a single vertex q in T. Note that $\bar{G}_i \leqslant \bar{H}_{i+3}$ by (4), hence $x, z \in \mathrm{Fix}(\bar{G}_i)$ and so every vertex of $[x, z]$ is fixed by $\bar{G}_i$. One can also observe that $b_i\bar{G}_ib_i^{-1} \leqslant \bar{H}_{i+2} \cap \bar{H}_{i+3}$, by (3), (4) and (5), therefore every vertex of $[y, z]$ is fixed by $b_i\bar{G}_ib_i^{-1}$. It follows that q is fixed by $\langle \bar{G}_i, b_i\bar{G}_ib_i^{-1}\rangle = \bar{H}_{i+3}$, as claimed. □

Proposition 3.2. *If $G_0 < G_1 < \dots$ is a good sequence of finitely generated groups, then the group L, constructed above, has property* (FA).

Proof. Suppose that L acts simplicially on a simplicial tree T without edge inversions. Let $d_T(\cdot,\cdot)$ denote the canonical path length metric on (the vertex set of) T. Observe that $\bar{G} := \bigcup_{i=0}^{\infty} \bar{G}_i$ is a subgroup of L (by (2)) and every element of this subgroup fixes a vertex of T by property (iii) from Definition 2.1 and part (a) of Remark 2.2. We need to consider two cases.

Case 1: $\mathrm{Fix}(\bar{G}) \neq \emptyset$. In this case, take any $x \in \mathrm{Fix}(\bar{G})$. Since T is a simplicial tree and $\mathrm{Fix}(\bar{H}_{i+2}) \neq \emptyset$ for every $i \in \mathbb{N}$, there are $n \in \mathbb{N}$ and $y \in \mathrm{Fix}(\bar{H}_{n+2})$ such that $\mathrm{d}_T(x, y)$ is minimal. Note that $x \in \mathrm{Fix}(\bar{G}_i)$ for all $i \in \mathbb{N}$, hence Lemma 3.1 implies that some vertex of $[x, y]$ is fixed by $\bar{H}_{n+3}$. It follows that $y \in \mathrm{Fix}(\bar{H}_{n+3})$, according to the choice of y. But now, in view of (5), we have that $y \in \mathrm{Fix}(L)$.

Case 2: $\mathrm{Fix}(\bar{G}) = \emptyset$; we will show that this case is impossible. Using [7, Theorem 4.12] one can find an infinite geodesic ray R in T such that every element of $\bar{G}$ fixes all but finitely many vertices of R. Without loss of generality we can assume that R starts at a vertex $x_1 \in \mathrm{Fix}(\bar{H}_4)$. Then $x_1 \in \mathrm{Fix}(\bar{G}_1)$ (by (4)). Since $\bar{G}_i$ is finitely generated it follows that for every $i \in \mathbb{N}$ the subgroup $\bar{G}_i$ fixes all but finitely many vertices of R. Let x_i denote the first vertex of R fixed by $\bar{G}_i$. Clearly $\bar{G}_i$ fixes all vertices of R after x_i and $x_i \in [x_1, x_{i+1}]$ for all $i \in \mathbb{N}$.

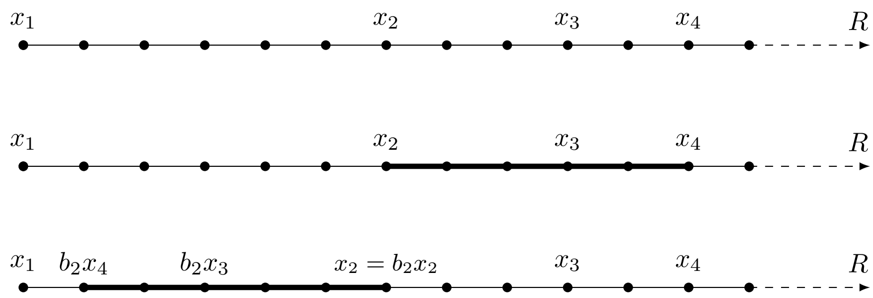

Figure 2. For $n = 1$, $\mathrm{d}_T(x_2, x_3) < \mathrm{d}_T(x_1, x_2)$.

Now, Lemma 3.1 yields that $[x_1, x_2] \cap \mathrm{Fix}(\bar{H}_5) \neq \emptyset$ and since $\bar{G}_2 \leqslant \bar{H}_5$ we see that x_2 is the first vertex of R that is fixed by $\bar{H}_5$. Arguing similarly, by induction, one shows that x_i is the first vertex of R which is fixed by $\bar{H}_{i+3}$, for all $i \in \mathbb{N}$.

Observe that if $x_i = x_{i+1}$ for some $i \in \mathbb{N}$, then $x_i \in \mathrm{Fix}(\bar{H}_{i+3}) \cap \mathrm{Fix}(\bar{H}_{i+4})$, which would imply that $x_i \in \mathrm{Fix}(L) \neq \emptyset$ by (5), contradicting to the assumptions of Case 2. Thus we can further suppose that the vertices $x_1, x_2, \dots$ are pairwise distinct.

Take any $n \in \mathbb{N}$. Let q denote the unique vertex of T that is equal to the intersection of the three geodesic segments $[x_n, x_{n+1}]$, $[x_{n+1}, b_{n+1} \circ x_{n+3}]$ and $[x_n, b_{n+1} \circ x_{n+3}]$, where $b_{n+1} \in \bar{H}_{n+4}$ is as above (in particular, $b_{n+1} \circ x_{n+1} = x_{n+1}$). Note that $b_{n+1} \circ x_{n+3} \in b_{n+1} \circ \mathrm{Fix}(\bar{G}_{n+3}) = \mathrm{Fix}(\bar{H}_{n+3})$ by (5); and so all of the vertices of the segment $[x_n, b_{n+1} \circ x_{n+3}]$ are fixed by $\bar{H}_{n+3}$. It follows that $q \in \mathrm{Fix}(\bar{H}_{n+3})$. But the only vertex of $[x_{n+1}, x_{n+3}]$ fixed by $\bar{G}_{n+3}$ is x_{n+3} by construction, hence the only vertex of $b_{n+1} \circ [x_{n+1}, x_{n+3}] = [x_{n+1}, b_{n+1} \circ x_{n+3}]$ fixed by $\bar{H}_{n+3} = b_{n+1} \bar{G}_{n+3} b_{n+1}^{-1}$ is $b_{n+1} \circ x_{n+3}$. Therefore $b_{n+1} \circ x_{n+3} = q \in [x_n, x_{n+1}]$ (see Figure 2). Consequently $b_{n+1} \circ [x_{n+1}, x_{n+3}] \subseteq [x_{n+1}, x_n]$, and as $x_{n+2} \in [x_{n+1}, x_{n+3}]$, $x_{n+2} \neq x_{n+3}$ we see that $b_{n+1} \circ x_{n+2} \in [x_{n+1}, x_n]$ and $b_{n+1} \circ x_{n+2} \neq x_n$. This yields that

$$\mathrm{d}_T(x_{n+1}, x_{n+2}) = \mathrm{d}_T(x_{n+1}, b_{n+1} \circ x_{n+2}) < \mathrm{d}_T(x_n, x_{n+1}).$$

Thus $\{\mathrm{d}_T(x_n, x_{n+1}) \mid n \in \mathbb{N}\}$ is an infinite strictly decreasing sequence of positive integers. Since this is impossible we must be in Case 1, and so L fixes some vertex of T. □

4. Constructing good sequences

In this section we suggest two approaches for constructing strictly ascending sequences of finitely generated groups $G_0 < G_1 < \dots$ satisfying conditions (i)-(iii) of Definition 2.1. The first method will use R. Thompson's group V, and the second method will be based on small cancellation theory over HNN-extensions.

4.1. **Construction using Thompson's group V.** R. Thompson's group V can be defined as the group of all piecewise linear right continuous bijections of the interval $[0, 1)$ which map dyadic rational numbers to dyadic rational numbers, are differentiable in all but finitely many dyadic rational numbers and such that on every maximal interval, where the function is linear, its slope is a power of 2. We refer the reader to [2] for a a good introduction to the group V.

The group V is finitely presented and simple [2]. Let every G_i be an isomorphic copy of V, $i = 0, 1, \dots$. To explain how G_i is embedded in G_{i+1}, consider the function $f : [0, 1] \to [0, 15/16]$ defined as follows:

$$f(x) := \begin{cases} x & \text{if } x \in \left[0, \frac{1}{4}\right) \\ \frac{x}{2} + \frac{1}{8} & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right) \\ 2x - 1 & \text{if } x \in \left[\frac{3}{4}, \frac{15}{16}\right) \\ x - \frac{1}{16} & \text{if } x \in \left[\frac{15}{16}, 1\right] \end{cases} .$$

Clearly f is continuous, increasing and piecewise linear on $[0,1]$. Also note that f induces a bijection between dyadic numbers on $[0,1]$ and $[0,15/16]$, and

$$f(1)=15/16,\ f(15/16)=7/8,\ f(7/8)=3/4 \text{ and } f(3/4)=1/2. \tag{6}$$

For each $i\in\mathbb{N}\cup\{0\}$ we define the embedding $\gamma_i : G_i \to G_{i+1}$ as follows: for any function $g\in V\cong G_i$ set

$$\gamma_i(g)(x) := \begin{cases} (f\circ g\circ f^{-1})(x) & \text{if } x\in \left[0,\frac{15}{16}\right) \\ x & \text{if } x\in\left[\frac{15}{16},1\right) \end{cases}.$$

Clearly $\gamma_i(V)\leqslant V_{[0,15/16)}$, where $V_S := \{h\in V \mid \operatorname{supp}(h)\subseteq S\}\leqslant V$ for any subset $S\subseteq[0,1)$ (where $\operatorname{supp}(h) := \{x\in[0,1)\mid h(x)\neq x\}$). It is also clear that γ_i is invertible and $\gamma_i^{-1} : V_{[0,15/16)}\to V$. Hence every γ_i is an isomorphism between V and $V_{[0,15/16)}$. Thus one can regard G_i inside of G_{i+1} as $V_{[0,15/16)}$ inside of V. Similarly, since we picked the function f to satisfy (6), G_{i-1} and G_i in G_{i+3} will correspond to $V_{[0,1/2)}$ and $V_{[0,3/4)}$ in V respectively.

Define $a_i\in G_{i+3}$ to be the element of V exchanging the intervals $[1/2,3/4)$ and $[3/4,1)$, which can be given by the following formula:

$$a_i(x) := \begin{cases} x & \text{if } x\in\left[0,\frac12\right) \\ x+\frac14 & \text{if } x\in\left[\frac12,\frac34\right) \\ x-\frac14 & \text{if } x\in\left[\frac34,1\right) \end{cases}.$$

Since $\operatorname{supp}(a_i)\cap[0,1/2)=\emptyset$, a_i will centralize $V_{[0,1/2)}$ in V, thus the condition (i) from Definition 2.1 is satisfied. Also observe that $a_iV_{[0,3/4)}a_i^{-1} = V_{[0,1/2)\cup[3/4,1)}$ in V and in order to establish (ii) we need to check that V is generated by $V_{[0,3/4)}$ and $V_{[0,1/2)\cup[3/4,1)}$. We will do this by showing that some generating set of V is contained in $\langle V_{[0,3/4)}, V_{[0,1/2)\cup[3/4,1)}\rangle$.

From [2, Lemma 6.1] we know that V is generated by its elements A,B,C and π_0, where

$$A(x) := \begin{cases} \frac{x}{2} & \text{if } x\in\left[0,\frac12\right) \\ x-\frac14 & \text{if } x\in\left[\frac12,\frac34\right) \\ 2x-1 & \text{if } x\in\left[\frac34,1\right) \end{cases},\quad B(x) := \begin{cases} x & \text{if } x\in\left[0,\frac12\right) \\ \frac{x}{2}+\frac14 & \text{if } x\in\left[\frac12,\frac34\right) \\ x-\frac18 & \text{if } x\in\left[\frac34,\frac78\right) \\ 2x-1 & \text{if } x\in\left[\frac78,1\right) \end{cases},$$

$$C(x) := \begin{cases} \frac{x}{2}+\frac34 & \text{if } x\in\left[0,\frac12\right) \\ 2x-1 & \text{if } x\in\left[\frac12,\frac34\right) \\ x-\frac14 & \text{if } x\in\left[\frac34,1\right) \end{cases},\quad \pi_0(x) := \begin{cases} \frac{x}{2}+\frac12 & \text{if } x\in\left[0,\frac12\right) \\ 2x-1 & \text{if } x\in\left[\frac12,\frac34\right) \\ x & \text{if } x\in\left[\frac34,1\right) \end{cases}.$$

One easily sees that $\pi_0\in V_{[0,3/4)}$, $C\circ\pi_0\in V_{[0,1/2)\cup[3/4,1)}$ and $\pi_0\circ B\circ\pi_0^{-1}\in V_{[0,1/2)\cup[3/4,1)}$. Note that $(A^{-1}\circ B)(x)=x$ for all $x\in[7/8,1)$, so taking any element $D\in V_{[0,1/2)\cup[3/4,1)}$ with $D([3/4,1)) = [7/8,1)$, we have $(D^{-1}\circ A^{-1}\circ B\circ D)(x)=x$ for all $x\in[3/4,1)$. Thus $D^{-1}\circ A^{-1}\circ B\circ D\in V_{[0,3/4)}$ and we have proved that $A,B,C,\pi_0\in\langle V_{[0,3/4)}, V_{[0,1/2)\cup[3/4,1)}\rangle$.

Hence the condition (ii) from Definition 2.1 holds for the sequence $G_0<G_2<\dots$. Finally, the condition (iii) from Definition 2.1 holds because V has property (FA), as shown by D. Farley in [11] (based on the notes of K. Brown).

4.2. **Construction using small cancellation methods.** Let us start with the following well-known observation (see [6, Remark on p. 680]).

Lemma 4.1. *Any countable group G can be embedded into a finitely generated group F with property* (FA). *If G is finitely presented then one can take F also to be finitely presented.*

Proof. Take any non-elementary word hyperbolic group H with property (FA) (e.g., a hyperbolic triangle group or a hyperbolic group with Kazhdan's property (T) if one looks for a torsion-free example). The group H is SQ-universal (as proved by T. Delzant [6, Thm. 3.5], and independently,

by A. Olshanskii [14, Thm. 1]), thus G can be embedded into some quotient F of H. Since property (FA) passes to quotients, this proves the first part of the lemma.

Now, suppose that G is finitely presented. This means that $G \cong \mathbb{F}_n/N$, where $\mathbb{F}_n$ is the free group of rank $n \geq 2$ and $N \lhd \mathbb{F}_n$ is a normal subgroup which the normal closure of finitely many elements $f_1, \dots, f_k \in \mathbb{F}_n$ for some $k \in \mathbb{N} \cup \{0\}$. According to [14, Theorems 2,3] the word hyperbolic group H contains a copy of $\mathbb{F}_n$ with the congruence extension property. Abusing the notation, let us identify $\mathbb{F}_n$ with this copy of it. The congruence extension property for $\mathbb{F}_n$ in H implies that $M \cap \mathbb{F}_n = N$, where M is the normal closure of $f_1, \dots, f_k$ in H. Therefore there is a natural embedding of $G \cong \mathbb{F}_n/N = \mathbb{F}_n/(\mathbb{F}_n \cap M)$ into $F := H/M$. As before, F will have (FA); F will also be finitely presented because it is a quotient of the finitely presented group H by the normal closure of finitely many elements. □

Remark 4.2. A more technical argument, still based on the methods from [14], would allow one to ensure in Lemma 4.1 that F is torsion-free if one starts with a torsion-free group G .

The construction of good sequences we suggest here employs the theory of small cancellation quotients of HNN-extensions, which was developed by G. Sacerdote and P. Schupp in [15]. We refer the reader to [15] or [13, V.11] for the details of this theory. This method is quite flexible and allows us to start with an *arbitrary* finitely generated group G_0. By Lemma 4.1 we can embed G_0 into a finitely generated group F_0 with property (FA), and we can let G_1 to be the free product of F_0 with an infinite cyclic group: $G_1 := F_0 * \langle u_1 \rangle$. For $i = 2, 3$ we proceed similarly: first embed G_{i-1} into a finitely generated group F_{i-1} with (FA) and then set $G_i := F_{i-1} * \langle u_i \rangle$. Now, suppose $G_1, G_2, \dots, G_{i+2}$ have already been built, with $i \in \mathbb{N}$. To simplify the notation we will identify each G_j, F_j and u_{j+1} with their canonical images in G_i, whenever $j < i$. Again, we embed G_{i+2} into a finitely generated group F_{i+2} with property (FA), and consider the HNN-extension E_{i+2}, of $F_{i+2} * \langle u_{i+3} \rangle$, defined by the presentation

$$E_{i+2} := \langle F_{i+2}, u_{i+3}, t_i \mid t_i g t_i^{-1} = g \text{ for all } g \in G_{i-1} \rangle. \tag{7}$$

Let $\{v_1, \dots, v_l\}$ be a finite generating set of $F_{i+2} * \langle u_{i+3} \rangle$. By construction the subgroup that G_{i-1} and $\langle u_i \rangle$ generate in F_{i+2} is isomorphic to their free product. Therefore

$$u_i^p G_{i-1} u_i^q \cap G_{i-1} = \emptyset \text{ whenever } p, q \in \mathbb{Z} \text{ and } p \neq -q. \tag{8}$$

Hence we can replace u_i with its power, if necessary, to assume that

$$u_i^{100j+100} v_j^{-1} u_i^{100j+1} \notin G_{i-1} \text{ for every } j = 1, \dots, l. \tag{9}$$

Now, consider the words $r_0, r_1, \dots, r_l$ defined by

$$r_0 := t_i^{-1} u_i t_i u_i^2 t_i^{-1} \dots u_i^{98} t_i u_i^{99} t_i^{-1} u_i^{100}, \text{ and}$$

$$r_j := v_j^{-1} u_i^{100j+1} t_i u_i^{100j+2} t_i^{-1} \dots u_i^{100j+98} t_i u_i^{100j+99} t_i^{-1} u_i^{100j+100}$$

for $j = 1, \dots, l$.

Observe that (8) and (9) imply that the words $r_0, \dots, r_l$ are cyclically reduced in the HNN-extension E_{i+2}. Let R be the set of all cyclically reduced conjugates of $r_0^{\pm 1}, \dots, r_l^{\pm 1}$. It is straightforward to check (using (8)) that for any two words $w_1, w_2 \in R$, representing distinct elements of E_{i+2}, at most three pairs of t_i-letters can cancel in the product $w_1 w_2$. It follows that the length of every piece relative to R is at most 7 and so R satisfies the small cancellation condition $\mathrm{C}'(1/6)$ (see [15]). Let $N \lhd E_{i+2}$ be the normal closure of R in E_{i+2} and let $G_{i+3} := E_{i+2}/N$. Then [15, Cor. 1] states that the natural epimorphism $\nu : E_{i+2} \to G_{i+3}$ is injective on $\langle F_{i+2}, u_{i+3} \rangle$. Letting $a_i := \nu(t_i) \in G_{i+3}$ and identifying G_{i-1} and G_i with their images in G_{i+3} we see that a_i centralizes G_{i-1} in G_{i+3} (by (7)). Moreover, since $r_j = 1$ in G_{i+3}, for every $j = 0, 1, \dots, l$, we see that the generating set $\{t, v_1, \dots, v_l\}$ of E_{i+2} is mapped inside of $\langle u_i, a_i u_i a_i^{-1} \rangle$ in G_{i+3}. Thus $G_{i+3} = \langle u_i, a_i u_i a_i^{-1} \rangle \leqslant \langle G_i, a_i G_i a_i^{-1} \rangle$.

Evidently, continuing this way we will obtain a strictly ascending sequence of finitely generated groups $G_0 < G_1 < \dots$ that satisfies the properties (i) and (ii) from Definition 2.1. It is also clear that this sequence satisfies property (iii) because for each $i \in \mathbb{N}$, $G_i < F_i < G_{i+1}$ and F_i has (FA).

5. Proof of the main result

Proof of Theorem 1.1. Using any of the procedures from the previous section we can construct a strictly ascending sequence $G_0 < G_1 < \dots$ of finitely generated groups with properties (i),(ii) and (iii). Section 2 tells us how to produce a sequence of finitely generated groups $L_1, L_2, \dots$ together with epimorphisms $\phi_i : L_i \to L_{i+1}$, $i \in \mathbb{N}$. Let $L := \lim_{i\to\infty}(L_i, \phi_i)$ be the direct limit of this sequence.

Observe that each L_i, $i \in \mathbb{N}$, splits as a non-trivial amalgamated free product, and so L_i has a fixed point-free action on the corresponding simplicial Bass-Serre tree T_i (cf. [16, Thm. I.4.7]). It follows that the limit L has a non-trivial action on an $\mathbb{R}$-tree T, which is the limit of the trees T_i, thus L does not have (F$\mathbb{R}$) (the $\mathbb{R}$-tree T can be constructed as in [9, Thm. 3.2] and non-triviality of the the induced action of L on T can be proved as in [8]; alternatively, the fact that L does not have (F$\mathbb{R}$) is a direct consequence of the general theorem of Culler and Morgan [5, Thm. 4.5], see also [18, Thm. 4.7]). On the other hand, the group L has property (FA) by Proposition 3.2.

For the second claim of the theorem, suppose that P is a finitely presented group that maps onto L. By a standard argument (see [4, Lemma 3.1]), there is $n \in \mathbb{N}$ such that P maps onto L_n. The idea is simple: P is a quotient of some free group $\mathbb{F}_m$ modulo a normal subgroup $N \lhd \mathbb{F}_m$, which is normally generated by finitely many elements. Any epimorphism from P to L gives rise to an epimorphism $\zeta : \mathbb{F}_m \to L$, which factors through each L_i. Let $N_i \lhd \mathbb{F}_m$ denote the kernel of the corresponding homomorphism $\zeta_i : \mathbb{F}_m \to L_i$. It follows that $N_i \leqslant N_{i+1}$ and $\ker\zeta = \bigcup_{i\in\mathbb{N}} N_i$. Evidently $N \leqslant \ker\zeta$, hence there is $l \in \mathbb{N}$ such that $N \leqslant N_i$ for all $i \geq l$ because N is the normal closure of finitely many elements of $\mathbb{F}_m$. Consequently, the homomorphism $\zeta_i : \mathbb{F}_m \to L_i$ factors through the natural homomorphism from $\mathbb{F}_m$ to $\mathbb{F}_m/N \cong P$ whenever $i \geq l$. Finally, since L_1 is finitely generated and $\zeta : \mathbb{F}_m \to L$ is surjective there is $k \in \mathbb{N}$ such that the homomorphisms $\zeta_i : \mathbb{F}_m \to L_i$ are surjective for all $i \geq k$. Thus one can take $n = \max\{k, l\}$. Therefore P will act non-trivially on the Bass-Serre tree T_n and so it does not have (FA). □

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